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THE INVERSION OF A SPHERICAL COMPRESSION WAVE IN FLUID*

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The spherically symmetric motion of fluid under the action of a spherical piston is considered. The method of merging asymptotic expansions in a small parameter which defines the low compressibility of fluid is used. The inversion of a compression wave is investigated. The point of inversion is determined and the asymptotics of solution in its neighborhood, which replaces the initial condition for shock wave construction, is obtained.

1. Statement of the problem. In the case of spherical symmetry the equations of motion and continuity, and those of Tait are of the form

$$\frac{\partial \bar u}{\partial \bar t}+\bar u\frac{\partial \bar u}{\partial \bar x}=-\frac{1}{\bar\rho}\frac{\partial \bar p}{\partial \bar x},\quad \frac{\partial \bar\rho}{\partial \bar t}+\bar u\frac{\partial \bar\rho}{\partial \bar x}+\bar\rho\left(\frac{\partial \bar u}{\partial \bar x}+\frac{2\bar u}{\bar x}\right)=0 \tag{1.1}$$

$$\bar p=A_1(\bar\rho/\bar\rho_0)^n+A_2,\quad \bar a_0=\sqrt{A_1 n/\bar\rho_0} \tag{1.2}$$

where $\bar\rho_0, \bar a_0$ are the fluid density and the speed of sound in the unperturbed medium.

We introduce the small parameter $\varepsilon=\bar p_0/(\bar a_0^2\bar\rho_0)$ using some characteristic pressure $\bar p_0$ and taking into account the low compressibility of fluid in the considered pressure range, we obtain for parameter ρ the formula $\bar\rho=\bar\rho_0(1+\varepsilon\rho)$ which, except for the constant, coincides with the dimensionless pressure.

Expanding the (expression for) pressure in (1.2) in series in small pressure variations, we obtain

$$\frac{d\bar p}{d\bar\rho}\frac{\bar\rho_0}{\bar p_0}=\frac{1}{\varepsilon}+k\rho,\quad k=n-1$$

In the dimensionless variables

$$u=\bar u\sqrt{\bar\rho_0/\bar p_0},\quad t=\bar t/T,\quad x=\bar x\sqrt{\bar\rho_0/\bar p_0}\,/T$$

the system (1.1) assumes the form

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{1+\varepsilon\rho}\frac{\partial \rho}{\partial x}(1+\varepsilon k\rho)\\ \varepsilon\frac{\partial \rho}{\partial t}+\varepsilon u\frac{\partial \rho}{\partial x}+(1+\varepsilon\rho)\left(\frac{\partial u}{\partial x}+\frac{2u}{x}\right)=0\end{gathered} \tag{1.3}$$

The usual kinematic condition apply at the piston $x=\varphi(t)$, and the initial conditions are homogeneous.

2. Construction of solution for $t\sim 1$. The problem formulated above is solved by the method of merging asymptotic expansions in ε /1,2/. Two zones are considered, viz. the external zone (subscript e) adjacent to the piston and the internal one (subscript i) adjacent to the leading perturbation front. Several expansion terms were obtained earlier for (intervals of) times of order unity, with continuous initial conditions that correspond to zero initial velocity of the piston.

In the external and internal zones the scales of variables, and the expansions are of the form

$$\begin{gathered}u=u_e=u_{e0}+\sqrt{\varepsilon}\,u_{e1}+\varepsilon u_{e2}+\varepsilon^{3/2}u_{e3}+\ldots\\ \rho=\rho_e=\rho_{e0}+\sqrt{\varepsilon}\,\rho_{e1}+\varepsilon\rho_{e2}+\varepsilon^{3/2}\rho_{e3}+\ldots,\quad x_e=x,\ t_e=t\end{gathered} \tag{2.1}$$

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$$u = \varepsilon u_i = \varepsilon\,(u_{i0} + \varepsilon u_{i1} + \varepsilon^{3/2} u_{i2} + \ldots) \tag{2.2}$$
$$\rho = \sqrt{\varepsilon}\,\rho_i = \sqrt{\varepsilon}\,(\rho_{i0} + \varepsilon\rho_{i1} + \varepsilon^{3/2}\rho_{i2} + \ldots),\quad x_i = x\sqrt{\varepsilon},\quad t_i = t$$

The first two terms external and two terms internal expansions were obtained in /2/ for $k = 0$. Below we set $k = 0$ which does not affect the principal expansion term. Merging (2.1) and (2.2) we obtain subsequent expansion terms. Omitting cumbersome operations, we present the final solutions for the external expansion

$$\begin{aligned}
u &= \frac{C}{x^2} + \varepsilon\left[\frac{C''}{2}\left(\frac{\varphi^2}{x^2} - 1\right) + \frac{2CC'}{x^2}\left(\frac{1}{\varphi} - \frac{1}{x}\right) + \right.\\
&\left.\frac{C^3}{2x^2}\left(\frac{1}{x^4} - \frac{1}{\varphi^4}\right)\right] + \varepsilon^{3/2}\frac{C'''}{3}\left(x - \frac{\varphi^3}{x^2}\right);\\
C &= \varphi^2(t)\,\varphi'(t),\qquad \varphi = \varphi(t)\\
\rho &= \frac{C'}{x} - \frac{C^2}{2x^4} - C''\sqrt{\varepsilon} + \varepsilon\left[\frac{C'''x}{2} - \frac{C'^2 + CC''}{2x^2} + \frac{9}{5}\frac{C^2C'}{x^5} - \right.\\
&\frac{3}{8}\frac{C^4}{x^8} - \frac{C}{x^4}\left(\frac{C''\varphi^2}{2} + \frac{2CC'}{\varphi} - \frac{C^3}{2\varphi^4}\right) - \frac{1}{x}\left(\frac{C'''\varphi^2 + 2C''\varphi'\varphi}{2} + \right.\\
&\left.\left.\frac{2C''^2 + 2CC''}{\varphi} - \frac{7}{2}C'\varphi'^2 + 2\varphi\varphi'^4\right)\right] + \varepsilon^{3/2}\left[-\frac{C^{(IV)}}{6}x^2 - \right.\\
&\frac{C^{(IV)}\varphi^3}{3x} - \frac{4C'''C}{3x} + \frac{CC'''\varphi^3}{3x^4} - \frac{C'C''}{x} + \frac{C''C^2}{2x^4} + 10\varphi'^5 - \\
&2\varphi'^3\varphi''\varphi - \frac{16C'\varphi'^3}{\varphi} + 2\varphi'C'\varphi'' - \frac{6C'C''}{\varphi} - \frac{C^{(IV)}\varphi^2}{2} - \\
&\left.C''\varphi\varphi'' + \frac{9}{2}\frac{C^2C''}{\varphi^4} - \frac{4CC'''}{\varphi} + \frac{7C''^2\varphi'}{\varphi}\right]
\end{aligned}$$

for the internal expansion

$$\begin{aligned}
u &= \sqrt{\varepsilon}\,\frac{C'}{x} + \frac{C}{x^2} + \varepsilon\left(\sqrt{\varepsilon}\,\frac{\Omega'}{x} + \frac{\Omega}{x^2}\right) + \varepsilon^{3/2}\left\{-\frac{C'^2}{4x^2} - \right.\\
&\frac{2CC'}{x^3\sqrt{\varepsilon}} + \frac{\sqrt{\varepsilon}}{2x}I(\xi) + \frac{C'C''}{x}\sqrt{\varepsilon}\ln(2x\sqrt{\varepsilon}) + \\
&\frac{C'^2}{2x^2}\ln(2x\sqrt{\varepsilon}) - \frac{1}{2x^2}\int_0^{\xi}\frac{C'^2(\tau)\,d\tau}{x\sqrt{\varepsilon} + t - \tau} + \\
&\left.\frac{1}{2x^2}\int_0^{\xi}F_1(\tau)\,d\tau + \frac{\sqrt{\varepsilon}}{2x}F_1(\xi) - \frac{C'^2 + CC''}{x^2}\right\}\\
\rho &= \frac{C'}{x} + \frac{\varepsilon}{x}\Omega' + \varepsilon^{3/2}\left\{\frac{C'C''}{x}\ln(2x\sqrt{\varepsilon}) - \frac{C'^2 + CC''}{x^2\sqrt{\varepsilon}} + \right.\\
&\left.\frac{C'^2}{4x^2\sqrt{\varepsilon}} - \frac{CC'}{x^3\varepsilon} - \frac{C^2}{2x^4}\varepsilon^{-3/2} - \frac{1}{2x}I(\xi) + \frac{1}{2x}F_1(\xi)\right\}\\
\Omega(\xi) &= \frac{C''\varphi^2}{2} + \frac{2CC'}{\varphi} - \frac{C^3}{2\varphi^4},\qquad \varphi = \varphi(\xi),\ \xi = -x\sqrt{\varepsilon} + t\\
F_1(\xi) &= -\frac{2C^{(IV)}\varphi^3}{3} - 4CC''' - 6C'C'' - \\
&2\int_0^{\xi}[C''^2(\tau) + C'C''']\ln(\xi - \tau)\,d\tau,\quad I(\xi) = \int_0^{\xi}\frac{C'^2(\tau)\,d\tau}{(x\sqrt{\varepsilon} + t - \tau)^2}
\end{aligned} \tag{2.3}$$

Conditions $C'(0) = 0,\ C''(0) = 0,\ C'''(0) = 0$ which follow from the requirement for the homogeneity of boundary condition at the leading characteristic were used in their derivation.

The inversion of a compression wave is due to nonlinear terms. The third term of expansion (2.3) for the internal zone, where the inversion takes place, is obtained from the nonlinear system (the two preceding ones are derived from the linear system). It is that term which shows the inhomogeneity of expansion (2.3) at exponentially large x owing to the presence of logarithmic terms. This indicates that at such distances the solution is of another form.

3. Construction of solutions for considerable time intervals. We seek a solution that is valid at the point of inversion, as well as after the shock wave formation. Note that at the leading perturbation front, be it a characteristic or a shock wave, the condition $u_s = \sqrt{\varepsilon}\rho_s$ is satisfied and is correct to exponentially small terms proportional to u_s^2, ρ_s^2. This is related to the exponentially small values of u_s and ρ_s at exponentially large distances, as will be seen from the solution obtained below.

Since the internal expansion, with which merging is taking place, was carried out for $t \sim 1$

which corresponds to $\xi \sim 1$, hence the solution for distant regions is derived for $\xi \sim 1$. Thus in that zone we have $t = x\sqrt{\varepsilon}$ which is correct to exponentially small terms.

When $\xi \sim 1$, expansion (2.3) yields for exponentially large distances the asymptotics

$$u = \rho\sqrt{\varepsilon} = x^{-1}\,[\sqrt{\varepsilon}\,C'(\xi) + \varepsilon^{3/2}\Omega'(\xi) + \varepsilon^2 C'(\xi)\,C''(\xi)\ln(2x\sqrt{\varepsilon}) + \varepsilon^2 F(\xi)/2 + \ldots] \tag{3.1}$$

since terms with powers of $1/x$ higher than the first are exponentially small in comparison with any of the terms in (3.1). Physically the condition

$$u = \rho\sqrt{\varepsilon} \tag{3.2}$$

is the consequence of the smallness of curvature of the spherical perturbation front at large distances. In the case of plane symmetry formula (3.2) is satisfied by the principal terms of expansion /2/.

To obtain the sought solution we pass in system (1.3) to coordinates $x,\ \xi = -x\sqrt{\varepsilon} + t$ and obtain for the principal expansion term the system

$$\frac{\partial u}{\partial \xi} - \sqrt{\varepsilon}\,\frac{\partial \rho}{\partial \xi} - u\sqrt{\varepsilon}\,\frac{\partial u}{\partial \xi} + u\,\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} = 0 \tag{3.3}$$

$$\frac{\partial u}{\partial \xi} - \sqrt{\varepsilon}\,\frac{\partial \rho}{\partial \xi} + \varepsilon u\,\frac{\partial \rho}{\partial \xi} - \sqrt{\varepsilon}\,u\,\frac{\partial \rho}{\partial x} - \frac{1}{\sqrt{\varepsilon}}\left(\frac{\partial u}{\partial x} + \frac{2u}{x}\right) = 0 \tag{3.4}$$

The principal expansion term is defined in the considered reqion by the equation

$$\frac{\partial u}{\partial \xi} - \sqrt{\varepsilon}\,\frac{\partial \rho}{\partial \xi} = 0$$

which, with allowance for conditions at the leading perturbation front come to relation (3.2) and the equation obtained by subtracting (3.3) from (3.4) reduce to condition (3.2). Taking into account (3.2) this equation assumes the form

$$\frac{\partial u}{\partial x} + \frac{u}{x} - \varepsilon u\,\frac{\partial u}{\partial x} = 0 \tag{3.5}$$

The relatively exponentially small term $\varepsilon u \partial u / \partial x$ is disregarded.

One of the exact solutions of that equation which contains the arbitrary function G is of the form

$$ux = G(\omega), \quad \omega = \xi + ux\varepsilon\ln x \tag{3.6}$$

If, however, the expansion is in ε, we have

$$ux = G(\xi) + \varepsilon G'(\xi)\,G(\xi)\ln x \tag{3.7}$$

Expansion (3.7) is unsuitable in the case of exponentially large x, and the second term in ω is then of the same order as the first.

Thus the principal expansion term in the case of considerable time intervals is obtained, when the nonlinear term is retained in Eq.(3.5). However, it is possible, using (3.6), to show that the order of that term is comparatively small everywhere, including the region where it must be taken into account. This is the essence of the cumulative effect of the inversion of a spherical compression wave in fluid.

Solution (3.6) evidently becomes expansion (2.2) from which asymptotics (3.1) are obtained when $G(\xi) = \sqrt{\varepsilon}C'(\xi)$.

Let us derive subsequent terms of expansion for the distant region. If that expansion is in powers of ε (and perhaps of $\ln\varepsilon$), system (3.3), (3.4) will show that relation (3.2) is satisfied by all approximations. This means that subsequent expansion terms are also satisfied by Eq.(3.5) and condition (3.2). Expanding the arbitrary function G in series, we obtain

$$\begin{aligned} G &= \alpha_0 G_0 + \alpha_1 G_1 + \ldots, \quad \alpha_i = \alpha_i(\varepsilon), \quad G_i = G_i(\omega) \\ ux &= \rho\sqrt{\varepsilon}x = \beta_0 v_0 + \beta_1 v_1 + \ldots, \quad \beta_i = \beta_i(\varepsilon) \end{aligned} \tag{3.8}$$

where v_i is related to G_j by formula (3.6).

Merging (3.8) with (3.1) we obtain for the distant zone an expansion of the form

$$ux = \rho\sqrt{\varepsilon}x = \sqrt{\varepsilon}C'(\omega) + \varepsilon^{3/2}\Omega'(\omega) + \varepsilon^2\ln\varepsilon\,\frac{C''(\omega)\,C'(\omega)}{2} + C^2(\omega)\left[\frac{F(\omega)}{2} + C'(\omega)\,C''(\omega)\ln 2\right] + \ldots \tag{3.9}$$

whose solution structure is such that for obtaining $u(x, t)$ it is necessary to solve it for u appearing in ω. A feature of the derived solution is that its order is lower than that of any asymptotic functions of α_i used in the expansion.

4. The inversion of a compression wave. The position of a shock wave is defined by the equation

$$\frac{dx_s}{dt} = \frac{1}{\sqrt{\varepsilon}} + \frac{u_s}{2} \qquad (4.1)$$

In variables $y = \ln x, v = ux, \xi_0 = \xi / \varepsilon$ along the characteristics of Eq. (3.5) $v = \text{const}$, and they are of the form

$$\xi_0 = -vy + F(v) \qquad (4.2)$$

When $F'(v) = y$, the characteristics intersect and a shock wave is formed.

Since we aim at obtaining a solution in the neighborhood of the point of inversion where v is low, we expand F at the instant when the shock wave is formed at the leading characteristic. We obtain

$$\xi_0 = -vy + k_1 v + k_2 v^2 \qquad (4.3)$$

If only linear terms with respect to v are taken into account in (4.3), the solution assumes the form

$$v = \xi_0 / (k_1 - y) \qquad (4.4)$$

and retention of the linear term yields

$$v = \{y - k_1 \pm [(y - k_1)^2 + 4k_2\xi_0]^{1/2}\} / (2k_2) \qquad (4.5)$$

Formulas (4.4) and (4.5) show that inversion occurs when $y = k_1$, i.e. when $k_1 > 0$.

In new variables Eq. (4.1) assumes the form

$$d\xi_0 / dy = -v / 2 \qquad (4.6)$$

The case of intersection of characteristics at the single point that represents their degenerate envelope, corresponds to solution (4.4). In Eq. (4.6) point ($\xi_0 = 0$, $v = 0$) is then a node, hence the integral curve cannot be uniquely selected. Moreover, when $y = k_1$ solution (4.4) generally does not exist at all times. This clearly shows that function $F(v)$ must necessarily be nonlinear and that it is necessary to use solution (4.5).

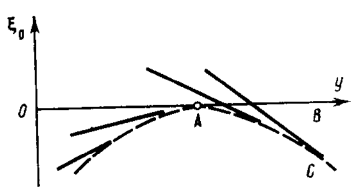

Fig.1

Depending on the sign of k_2, solution (4.3) substantially varies in the neighborhood of the point of inversion. When $k_2 < 0$, the envelope of characteristics begins in the zone $\xi_0 > 0$, i.e. initial inversion does not occur at the leading characteristic. The characteristics and their envelope $y = k_1 + \sqrt{-4k_2\xi_0}$ (shown by the dash line) in the ξ_0, v plane are plotted in Fig.1 for $k_2 > 0$.

Let us consider the case of $k_2 > 0$ and explain the relation between solutions (4.4) and (4.5).

For solution (4.5) to be formally transformed to (4.4), when $k_2\xi_0 / (y - k_1)^2 \ll 1$, it is necessary to use the plus sign in (4.5) for $y < k_1$ and the minus sign for $y > k_1$. However the stipulation of uniqueness of solution allows the characteristics to extend only up to the tangency point on the envelope (to zone ABC, Fig.1). Analytically this means that for $y > k_1$ formula (4.5) must be taken with the plus sign.

It has, thus, been proved that the linearization of function $F(v)$ provides an asymptotically correct result only up to the point of inversion, beyond which it is necessary to take into account the nonlinearity of function F. This can also be explained by the geometry of behavior of characteristics.

Using (4.3) it is possible to reduce Eq. (4.4) to the form

$$\frac{dv^2}{d\xi_0} = -\frac{2v^2}{\xi_0 + k_2 v^2} \qquad (4.7)$$

which shows that the inversion point ($\xi_0 = 0, v^2 = 0$) is a saddle when Eq. (4.7) is expressed in variables ξ_0, v^2. The only suitable integral curve is

$$v_s = (-3\xi_{0s} / k_2)^{1/2} \qquad (4.8)$$

In the neighborhood of the inversion point from (4.8) and (4.6) for the dependence of the shock on x_s we obtain

$$u_s = \rho_s \sqrt{\varepsilon} = \frac{3}{4} \frac{\ln x_s - k_1}{k_2 x_s} \qquad (4.9)$$

Let us determine coefficients k_1 and k_2. Since we aim at the determination of the first two terms of (2.2), it is sufficient to stipulate only the fulfillment of conditons $C(0) = C'(0) = C'''(0) = 0$. Hence for small time intervals

$$\varphi(t) = x_0 + \gamma_3 t^3 / 3! + \gamma_5 t^5 / 5! + \gamma_6 t^6 / 6! + \ldots$$

which shows that $F(v) = G^{-1}(v) / \varepsilon$, where G^{-1} is the inverse of function G. Hence the linearization of $F(v)$ is equivalent to the linearization of $C'(\omega)$. When v is small, let us determine the inverse function of G. Using (3.9) we obtain

$$k_1 = \varepsilon^{-3/2} / (\gamma_3 x_0^2), \; k_2 = -(34\gamma_3^2 + \gamma_6 x_0) / (4\varepsilon x_0^3 \gamma_3^3)$$

Obviously $k_1 > 0$ corresponds to $\gamma_3 > 0$, i.e. to a compression wave. The case of $k_2 > 0$ (or $\gamma_6 < -34\gamma_3^2 / x_0$) corresponds to a weakened compression wave due to the respective of φ with respect to t (as compared with those γ_6 at which discontinuity occurs earlier and not on the leading characteristic). At the inversion point we obtain the expression $\exp[1 / (\varepsilon^{3/2}\gamma_3 x_0^2)]$.

To determine the shock wave behavior outside the neighborhood of the inversion point it is necessary to solve Eq.(4.6) in which $v(\xi_0, y)$ is obtained from (4.2), and to substitute asymptotics (4.9) for the initial condition.

The constructed solution can be obtained using the method proposed in /3/.

With certain constraints on the law of piston motion at large distances from the inversion point the transformation to asymptotics first proposed by Landau is realized.

The equation which enables us to obtain the above asymptotics is of the form

$$dv_s/dy_s = 0.5 v_s \{[\varepsilon^{3/2} C'' (C'^{-1} (v_s/\sqrt{\varepsilon}))]^{-1} - y_s\}^{-1} \tag{4.10}$$

where C'' is obtained from the respective argument, and C'^{-1} is the inverse function of C'. In the wide class of problems with large y_s the expression in brackets in (4.10) can be neglected as small in comparison with y_s. The Landau asymptotics is then valid.

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